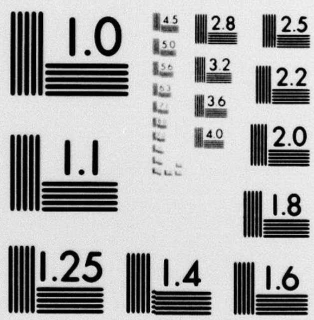
AD-A078 959 PRINCETON UNIV NJ DEPT OF STATISTICS F/G 12/1
AN IMPLICIT FUNCTION THEOREM FOR GROUP EQUATIONS GENERATED BY A--ETC(U)
AUG 79 P R THRIFT N00014-79-C-0322
UNCLASSIFIED TR-150-SER-2 NL

1 OF 1
AD
A078959

END
DATE
FILMED
1-80
DDC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

AD A078959

LEVEL II

6

AN IMPLICIT FUNCTION THEOREM FOR GROUP EQUATIONS
GENERATED BY A FINITE AUTOMATON.

by

Philip R. Thrift
Princeton University

DDC
RECEIVED
DEC 17 1979
E

TR-
Technical Report No. 150, Series 2
Department of Statistics
Princeton University
August 1979

12 9

Research supported in part by a contract with the Office of Naval Research, No. N00014-79-C-0322 awarded to the Department of Statistics, Princeton University, Princeton, New Jersey.

DDC FILE COPY

THIS DOCUMENT IS BEST QUALITY PRACTICABLE.
THE COPY FURNISHED TO DDC CONTAINED A
SIGNIFICANT NUMBER OF PAGES WHICH DO NOT
REPRODUCE LEGIBLY.

This document has been approved
for public release and sale; its
distribution is unlimited.

406873

ABSTRACT

The study of mathematical semantics involves the study of mappings from a formal language L to a configuration space C. In this paper we consider the special case where $L = L(\mathcal{D})$, $\mathcal{D}$ is a finite automaton, C is a group G, and the semantic map is defined by associating each production of $\mathcal{D}$ with a group element of G, $p \rightarrow \theta(p)$. A sequence of productions would then be mapped to a product of corresponding group elements. When the semantic map is observed on sentences in $L(\mathcal{D})$, we discuss a graph-theoretic method of solving for θ.

D =

approaches theta (P)

theta

Accession For
NTIS GRA&I ☑
DDC TAB ☐
Unannounced ☐
Justification
By
Distribution/
Availability Codes
Dist | Avail and/or special
A | 23

05

page - A -

1. INTRODUCTION

The study of mathematical semantics [3] involves the study of mappings from a formal language L to a configuration space C. (The reader is referred to [2] and [5] for pattern and automata theoretic notions.) In some of the cases that have been investigated, we can consider the set C as a set of operations G on some set of objects. The algebraic nature of these operations and sets of objects can be very simple or complex, depending on a particular application. In [3] these operations are called connectors when they deal with establishing or breaking relationships between these objects.

In the case where $L = L(\mathcal{D})$, $\mathcal{D}$ = a deterministic finite state diagram (or wiring diagram for some finite state automaton) over some alphabet Σ, and G (the set of operations) is a group, then we can define a semantic map by first associating an element of G with each production $p: i \xrightarrow{x} j$ of $\mathcal{D}$, where $x \in \Sigma$ and i, j are states of $\mathcal{D}$. If we write this map as $p \longrightarrow \theta(p) \in G$, p a production of $\mathcal{D}$, then for each sentence $x_1 \ldots x_k$ in $L(\mathcal{D})$, we define

$$\Phi(\theta)(x_1 \ldots x_k) = \theta(p_1) \ldots \theta(p_k) \quad \text{(multiplication in G)},$$

where

$$S \xrightarrow{x_1} i_1 \xrightarrow{x_2} i_2 \ldots i_{k-1} \xrightarrow{x_k} i_k ,$$

$$p_j: i_{j-1} \xrightarrow{x_j} i_j \quad \text{are productions of } \mathcal{D} ,$$

S = start state, and $i_k \in$ final state set F of $\mathcal{D}$. Thus $\Phi(\theta)$ is a semantic map $\Phi(\theta): L(\mathcal{D}) \rightarrow G$. A semantic map $\gamma: L(\mathcal{D}) \rightarrow G$ is called sequential if there is some map θ, $p \rightarrow \theta(p)$ p a production of $\mathcal{D}$, such that $\gamma = \Phi(\theta)$.

In this paper we consider the following problem: suppose the structure of the deterministic finite automaton $\mathcal{D}$ is known and we can observe $\gamma(w)$, $w \in L(\mathcal{D})$. Under the assumption that γ is sequential, what can we say about all possible solutions of $\Phi(\theta) = \gamma$? The answer is stated in the form of an implicit function theorem and a procedure is given for calculating solutions.

2. FINITE STATE DIAGRAMS AND GRAPH-THEORETIC NOTIONS

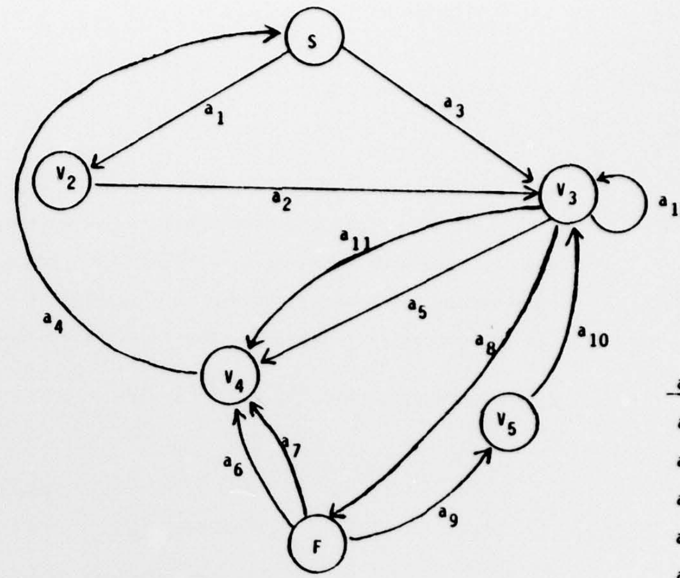
Let F be a directed pseudograph with vertex set

$V(F) = \{v_1, \ldots, v_p\}$ and arc set

$A(F) = \{a_1, \ldots, a_q\}$. (See [4] for graph-theoretic notions.) To each arc a_j corresponds an ordered pair $(\alpha(a_j), \beta(a_j))$, $\alpha(a_j), \beta(a_j) \in V(F)$. If $\alpha(a_j) = \beta(a_j)$ then we call a_j a loop. If Σ is some finite alphabet $\{a, b, c, \ldots\}$ and there is a map $A(F) \xrightarrow{\sigma} \Sigma$ satisfying the property: $\alpha(a_j) = \alpha(a_k)$, $j \neq k \longrightarrow \sigma(a_j) \neq \sigma(a_k)$, and if S is a single vertex of $V(F)$, F a (non-empty) subset of $V(F)$, then $\mathcal{D} = (F, \Sigma, \sigma, S, F)$ is called a finite state diagram with state set $V(F)$ and productions $\alpha(a_j) \xrightarrow{\sigma(a_j)} \beta(a_j)$, $j = 1, \ldots, q$. An example appears in Figure 1 below.

In this paper we shall consider, for simplicity, only the case where F is also a single state of $V(F)$ and $S \neq F$, so that $p = |V(F)| \geq 2$.

A finite state diagram can be used to generate sentences in Σ^* (that is, sequences $x_1 \ldots x_k$ from Σ). To generate a sentence in Σ^* we take a walk $a_{i_1} \ldots a_{i_k}$ from S to F in F (that is, a sequence of arcs such that $S = \alpha(a_{i_1})$, $\beta(a_{i_j}) = \alpha(a_{i_{j+1}})$ $j = 1, \ldots, k-1$, $\beta(a_{i_k}) = F$). We then produce the sentence



a	σ(a)
a_1	a
a_2	b
a_3	b
a_4	a
a_5	b
a_6	a
a_7	b
a_8	d
a_9	c
a_{10}	a
a_{11}	c
a_{12}	a

Figure 1.

$\sigma(a_{i_1} \dots a_{i_k}) = \sigma(a_{i_1}) \dots \sigma(a_{i_k})$ by concatenation. The language generated by D is defined to be

$$L(D) = \{w \in \Sigma^* | w = \sigma(a_{i_1} \dots a_{i_k}) \text{ for some walk } a_{i_1} \dots a_{i_k} \text{ from } S \text{ to } F\}.$$

$L(D)$ is called a <u>finite state language</u>.

In addition, given D, and any $w \in L(D)$ we can obtain the (unique) walk $a_{i_1} \dots a_{i_k}$ that generated w by using the uniqueness property of σ.

It may happen that there are no walks from S to F, in which case $L(D) = \phi$. However, we shall assume that F has the following connectedness property: for every $v_j \in \{v_1, \dots, v_p\}$, there is a walk $a_{i_1} \dots a_{i_k}$ from S to F that "passes through" v_j. It is apparent if a given state diagram D has a vertex v_j that does not have this property, then it can be eliminated from D (along with any incident arcs) and $L(D)$ will remain the same.

Suppose we list the vertices $V(F)$ as $\{S, v_2, \dots, v_{p-1}, F\}$. V_S is defined to be the following subset of V: $S \in V_S$ and also any $v_j \in \{v_2, \dots, v_{p-1}\}$ is in V_S if there is some <u>path</u> from S to v_j (that is, a walk from S to v_j with no vertex repeated) that does not pass through F. V_F is defined to be $V(F) - V_S$, so that $V(F) = V_S \cup V_F$. Note that neither V_S nor V_F are ϕ since $S \in V_S$ and $F \in V_F$. Note also that elements v_k in $V_F - \{F\}$ have the property that any path from S to v_k passes through F.

For any directed pseudograph F, a subgraph T is called a <u>directed tree</u> if:

(a) There is exactly one vertex of T called the <u>root</u> which no arc of T enters.

(b) Every vertex of T except the root has exactly one entering arc.

(c) There is a unique path from the root to each vertex in T.

If $V(T)$ is the vertex set of T, then we say that T spans $V(T)$.

We now wish to construct two directed trees in F: T_S rooted at S and spanning V_S, and T_F rooted at F and spanning V_F.

We construct T_S as follows. Start with S and define $V_S^{(0)} = S$. Let $V_S^{(1)}$ be the set of all vertices in $V_S - V_S^{(0)}$ that can be reached from $V_S^{(0)}$ in a single arc. Let $A_S^{(1)}$ be a set of arcs starting from $V_S^{(0)}$ and having the property that every element of $V_S^{(1)}$ is reached by exactly one arc of $A_S^{(1)}$. In general, let $V_S^{(k)}$ be the set of vertices in $V_S - V_S^{(1)} - \dots - V_S^{(k-1)}$ that can be reached from a vertex of $V_S^{(k-1)}$ in a single arc, and $A_S^{(k)}$ a set of arcs starting from $V_S^{(k-1)}$ and having the property that every element of $V_S^{(k)}$ is reached by exactly one arc of $A_S^{(k)}$. Continue until $V_S^{(k)}$ is empty. Let $T_S = \{V_S^{(0)} \cup V_S^{(1)} \cup \dots, A_S^{(1)} \cup A_S^{(2)} \cup \dots\}$. Clearly, T_S is a tree.

<u>Proposition</u>. T_S spans V_S.

<Proof>. Let $v_j \in V_S$. If $v_j = S$ then we are done. Otherwise, suppose $S \xrightarrow{b_1} v_{j_1} \xrightarrow{b_2} \dots \longrightarrow \dots \xrightarrow{b_k} v_{j_k} = v_j$ where

$v_{j_1}, \ldots, v_{j_k}$ are also in V_S. Then

$$v_{j_1} \in V_S^{(0)} \cup V_S^{(1)}$$
$$\vdots$$
$$v_{j_k} \in V_S^{(0)} \cup \ldots \cup V_S^{(k)}$$

can be proven inductively. ||

We construct T_F in exactly the same manner (substituting F for S).

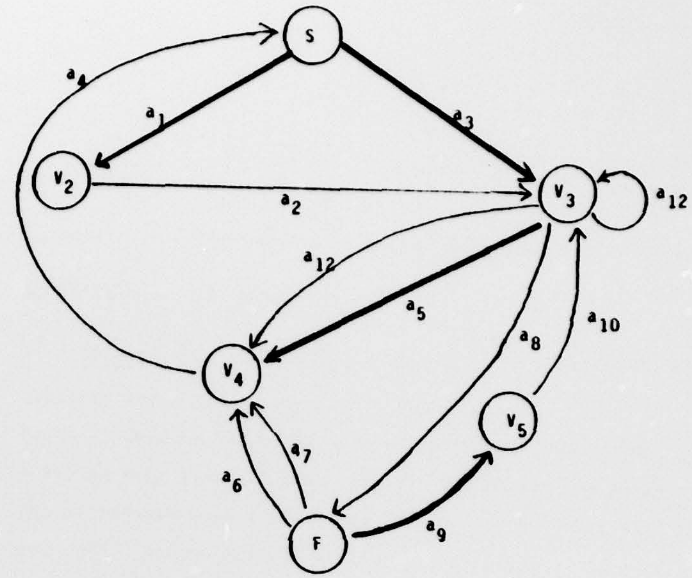
<u>Proposition</u>. T_F spans V_F.

⟨Proof⟩. Let $v_k \in V_F$. Then there is a path

$$F \xrightarrow{c_1} v_{k_1} \xrightarrow{c_2} \ldots \xrightarrow{c_j} v_{k_j} = v_k$$

where $v_{k_1}, \ldots, v_{k_{j-1}}$ are also in V_F, as is apparent from the definition of V_F. The proof then follows exactly as before. ||

In our previous example, we have



$A(T_S) = \{a_1, a_3, a_5\}$
$A(T_F) = \{a_9\}$

<u>Figure 2.</u>

3. SEMANTIC MAPS AND AN IMPLICIT FUNCTION THEOREM

We now consider a group G and a map

$$\theta: A(F) \to G .$$

Since there are q arcs in $A(F)$ we shall consider θ as an element of G^q with $\theta_j = \theta(a_j)$ $j = 1, \ldots, q$ called the arc values. Let Γ be the set of all semantic maps $\gamma: L(D) \to G$ and let

$$\phi: G^q \to \Gamma \quad \text{be defined by}$$

$$\phi(\theta)w = \theta(a_{i_1})\,\theta(a_{i_2}) \ldots \theta(a_{i_k}) \quad \text{if } a_{i_1}, \ldots, a_{i_k}$$

is the walk from S to F determined by the sentence w. Let us suppose the arcs are listed so that $a_1, \ldots, a_{p-2}$ are the arcs in $T_S \cup T_F$, with $a_{p-1}, \ldots, a_q$ remaining. Let $r = p-2$ and $m = q-r$. We now state the following implicit function theorem.

Theorem. Assume that $\gamma_0 \in \Gamma$ is sequential. Then there exist functions $\phi_1, \ldots, \phi_m$: $G^r \to G$ (depending, of course, on γ_0) such that

$$\{\theta \in G^q: \phi(\theta) = \gamma_0\} =$$

$$\{\theta \in G^q: (\theta_1, \ldots, \theta_r) \in G^r, \theta_{r+1} = \phi_1(\theta_1, \ldots, \theta_r) \text{ for } 1=1,\ldots,m\}.$$

Remark. In the case where $r = 0$ (that is, the case where T_S and T_F consist only of single vertices S and F) then there is a unique solution of $\phi(\theta) = \gamma_0$.

<Proof>. By assumption there is some $\hat{\theta} \in G^q$ satisfying $\phi(\hat{\theta}) = \gamma_0$. For each $(\theta_1, \ldots, \theta_r) \in G^r$ we shall first show that there exists a $(\theta_{r+1}, \ldots, \theta_q) \in G^m$ such that $\phi(\theta) = \gamma_0$, and then show that $(\theta_{r+1}, \ldots, \theta_q)$ is uniquely determined. The uniqueness argument gives a procedure for constructing the maps $\phi_1, \ldots, \phi_m$, given γ_0.

Let v_j be an element of $V_S^{(1)}$. Then v_j appears in F as

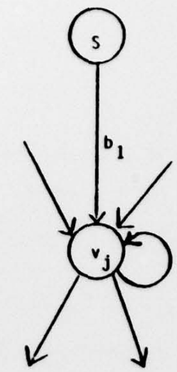


Figure 3.

with incoming arcs $b_1, \ldots, b_r$, loops $l_1, \ldots, l_t$, and outgoing arcs $c_1, \ldots, c_s$. Assume that b_1 is the arc that appears in T_S. Note that $\{c_1, \ldots, c_s\}$ is not empty if $A(T_S)$ is not empty. Update the arc values by the following system of assignments:

$$\hat{\theta}(b_i) \longleftarrow \hat{\theta}(b_i)\,\hat{\theta}(b_1)^{-1}\,\theta(b_1) \qquad i = 1, \ldots, r$$

$$\hat{\theta}(c_j) \longleftarrow \theta(b_1)^{-1}\,\hat{\theta}(b_1)\,\hat{\theta}(c_j) \qquad j = 1, \ldots, s$$

$$\hat{\theta}(l_k) \longleftarrow \theta(b_1)^{-1}\,\hat{\theta}(b_1)\,\hat{\theta}(l_k)\,\hat{\theta}(b_1)^{-1}\,\theta(b_1) \qquad k = 1, \ldots, t .$$

Note that the arc value of b_1 is changed to $\theta(b_1)$. This operation is performed for each vertex v in $V_S^{(1)}$ and then repeated for $V_S^{(2)}, V_S^{(3)}, \ldots$ until the vertex set of V_S is exhausted, and noting that once an arc value of an arc of T_S is set to the appropriate

group element, it is not changed by any subsequent update procedure.

We then proceed to the vertex set V_F and perform the same operations. Note that after any one of these update operations is performed, the value of ϕ remains the same (that is, the semantic map does not change). The result of these updating procedures is an element of G^q $\theta = (\theta_1, \ldots, \theta_r, \theta_{r+1}, \ldots, \theta_q)$ satisfying $\phi(\theta) = \gamma_0$.

We now proceed to show that $\theta_{r+1}, \ldots, \theta_q$ are unique in a constructive manner. Begin at F and consider all arcs entering F. Let $\{g_1, \ldots, g_r\}$ be the arcs entering F from V_S and $\{h_1, \ldots, h_s\}$ be the arcs entering F from V_F. Now $\{g_1, \ldots, g_r\}$ is not empty from the assumed connectedness property satisfied by F. Since g_1 comes from a vertex in V_S, it is connected to T_S, and thus there is some walk $a_{i_1} a_{i_2} \ldots a_{i_k} g_1$ from S to F with $a_{i_1} \ldots a_{i_k}$ a path in T_S. Therefore

$$\theta(a_{i_1}) \ldots \theta(a_{i_k})\, \theta(g_1) = \gamma_0(w)$$

or

$$\theta(g_1) = \theta(a_{i_k})^{-1} \ldots \theta(a_{i_1})^{-1}\, \gamma_0(w)$$

where w is the sentence in $L(D)$ corresponding to $a_{i_1} \ldots a_{i_k} g_1$. This calculation is performed for each $i = 1, \ldots, r$.

Now consider an h_j. Since h_j comes from a vertex in V_F, it is connected to T_F, and thus there is some walk $b_{j_1} \ldots b_{j_l} h_j$ from F to F with $b_{j_1} \ldots b_{j_l}$ a path in T_S. Therefore

$$\theta(a_{i_1}) \ldots \theta(a_{i_k})\, \theta(g_1)\, \theta(b_{j_1}) \ldots \theta(b_{j_l})\, \theta(h_j) = \gamma_0(u)$$

where u is the sentence corresponding to $a_{i_1} \ldots a_{i_k} g_1 b_{j_1} \ldots b_{j_l} h_j$. We can then solve uniquely for $\theta(h_j)$.

Note that for each arc in $\{g_1, \ldots, g_r\}$ and $\{h_1, \ldots, h_s\}$ we provided a sentence w or u that contained that particular arc from which $\theta(g_i), \theta(h_j)$ was recovered.

Let us now consider any arc a which is not in $A(T_S) \cup A(T_F)$ and show that $\theta(a)$ can be recovered uniquely from a single sentence. Let $C^{(j)} = \{\text{arcs } a \text{ of } F \mid d(\beta(a), F) = j\}$, that is the distance (the number of arcs in the shortest path) from $\beta(a_k)$ to F is j. Now $\{g_1, \ldots, g_r\} \cup \{h_1, \ldots, h_s\} = C^{(0)}$, and $\theta(g_1), \ldots, \theta(g_r), \theta(h_1), \ldots, \theta(h_s)$ have been computed.

Assume that we have computed $\theta(a)$ for arcs in $C^{(k)}$, $k < j$. Let a be an arc not in $A(T_S) \cup A(T_F)$ but in class $C^{(j)}$. Let $a\, a_{i_1} \ldots a_{i_j}$ be a shortest path from $\beta(a)$ to F. Now since $\alpha(a)$ is either in V_S or V_F we proceed exactly as before plus use the induction hypothesis to solve uniquely for $\theta(a)$. In this case also, a sentence w is produced from which $\theta(a)$ is calculated. Since by the connectedness property satisfied by F, a is in one of the sets $C^{(0)}, C^{(1)}, \ldots$ ||.

4. CONCLUSION

A rather vague statement of the implicit function theorem in Euclidean space (Fleming [1]) is that

"if $f(x) = 0$ consists of m independent equations $f_1(x) = 0, \ldots, f_m(x) = 0$ where $x \in E^q$ and $1 \le m < q$, then m of the variables of x are determined implicitly in terms of the other $q - m$."

Although the equation $\phi(\theta) = \gamma_0$ appears to involve an infinite number of group equations $\phi(\theta)w = \gamma_0(w)$ (one for each sentence in $L(D)$), if the finite set of sentences generated in the above proof

is $w_1, \ldots, w_m$, then the finite number of equations

$$\phi(\theta)w_j = \gamma_0(w_j) \qquad j = 1, \ldots, m$$

determine $\theta_{r+1} = \phi_1(\theta_1, \ldots, \theta_r)$ for $1 = 1, \ldots, m$ in the statement of the theorem. Also, any additional sentences will not reduce the solution set. In this sense, we can say that $\phi(\theta) = \gamma_0$ involves m "independent" group equations.

ACKNOWLEDGEMENT

The author wishes to express his gratitude to Professor Ulf Grenander for discussions on the topic in this paper.

REFERENCES

[1] Fleming, W. H. (1965). Functions of Several Variables. Addison-Wesley.

[2] Grenander, U. (1976, 1978). Lectures in Pattern Theory: Vol. 1 - Pattern Synthesis, Vol. 2 - Pattern Analysis. Springer-Verlag.

[3] ____________ (July 1978). "On Mathematical Semantics: A Pattern Theoretic View." Reports in Pattern Analysis No. 71, Division of Applied Mathematics, Brown University, Providence, Rhode Island.

[4] Harary, F. (1969). Graph Theory. Addison-Wesley.

[5] Hopcroft, J. E. and Ullman, J. D. (1969). Formal Languages and Their Relation to Automata. Addison-Wesley.

THIS DOCUMENT IS BEST QUALITY PRACTICABLE.
THE COPY FURNISHED TO DDC CONTAINED A
SIGNIFICANT NUMBER OF PAGES WHICH DO NOT
REPRODUCE LEGIBLY.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER T.R.#150, Series 2	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) AN IMPLICIT FUNCTION THEOREM FOR GROUP EQUATIONS GENERATED BY A FINITE AUTOMATON		5. TYPE OF REPORT & PERIOD COVERED Technical Report
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Philip R. Thrift		8. CONTRACT OR GRANT NUMBER(s) N00014-79-C0322
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Statistics Princeton University Princeton, N.J.		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research (code 436) Arlington, Virginia 22217		12. REPORT DATE August 1979
		13. NUMBER OF PAGES 13
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

79 12 14 077

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The study of mathematical semantics involves the study of mappings from a formal language L to a configuration space C. Here we consider the special case where $L = L(\mathcal{D})$, $\mathcal{D}$ is a finite automaton, C is a group G, and the semantic map is defined by associating each production of $\mathcal{D}$ with a group element of G,$p \to \theta(p)$. A sequence of productions would then be mapped to a product of corresponding group elements. When the semantic map is observed on sentences in $L(\mathcal{D})$, we discuss a graph-theoretic method of solving for θ.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

S/N 0102-014-6601

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

DISCLAIMER NOTICE

THIS DOCUMENT IS BEST QUALITY PRACTICABLE. THE COPY FURNISHED TO DDC CONTAINED A SIGNIFICANT NUMBER OF PAGES WHICH DO NOT REPRODUCE LEGIBLY.